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RECURSIVELY GENERATED STURM-LIOUVILLE POLYNOMIAL SYSTEMS

A THESIS

Presented to

The Faculty of the Graduate Division

by

John William Jayne

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

in the School of Mathematics

Georgia Institute of Technology

December, 1965

RECURSIVELY GENERATED STURM-LIOUVILLE POLYNOMIAL SYSTEMS

Approved:

Chairman

Date approved by Chairman: Nov. 23, 1965

ACKNOWLEDGMENTS

This thesis evolved from an attempt to answer some questions raised by Dr. M. B. Sledd concerning a certain sequence of recursively generated polynomials. The investigation prompted by those and related questions has been carried out by Dr. Sledd, Dr. D. V. Ho, and myself; some of the results presented here are to appear in the Duke Mathematical Journal in a joint paper having the same title as this thesis.

Working under the direction of Dr. Sledd and Dr. Ho has been in many ways a rewarding endeavor for me, and I wish to express my appreciation to them. I also wish to thank Dr. H. A. Gersch of the School of Physics for consenting to read the thesis, and to acknowledge my considerable indebtedness to the graduate faculty of the School of Mathematics for their contribution to my education.

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CHAPTER I

INTRODUCTION

A three-term recurrence relation

$$\begin{aligned} \varphi_o(x) &= 1 \\ \varphi_1(x) &= A_o x + B_o \\ \varphi_{n+1}(x) &= (A_n x + B_n)\varphi_n(x) - C_n \varphi_{n-1}(x) , \end{aligned} \tag{1}$$

where $A_o \neq 0$ and $A_n C_n \neq 0$ $(n = 1,2,3,...)$, generates a sequence $\{\varphi_n(x)\}$ of polynomials in which φ_n is of degree exactly n. Some (but not all) sequences so generated are Sturm-Liouville polynomial systems -- that is, sequences $\{\varphi_n(x)\}$ of polynomials in which, for each n, the nth-degree polynomial φ_n is a solution of a differential equation of the form

$$a_o(x)y'' + a_1(x)y' + [a_2(x) + \lambda_n]y = 0, \tag{2}$$

where λ_n is a parameter depending on n but not on x [1]. Since much is known about Sturm-Liouville systems (the Legendre polynomials $P_n(x)$, for example), and since (1) is of some interest [2,3,4,5,6,8,9,10], it would be useful to have

(i) a straightforward procedure for constructing the differential equation (2) from the coefficients in (1) under the <u>supposition</u> that the polynomials generated by (1) are solutions of (2);

(ii) a simple criterion for deciding, once (2) is constructed, whether the polynomials generated by (1) <u>actually are</u> solutions of (2).

In Chapter II such a procedure and such a criterion are deduced, and in each only the coefficients A_n, B_n, C_n in (1) are used. In addition, it is shown that if $\lambda_i \neq \lambda_j$ $(i \neq j;\ i,\ j = 0,1,2,\ldots)$, then the criterion has a relatively simple form.

A study of the polynomials which satisfy the criterion of Chapter II is undertaken in Chapter III. Initially the restriction $\lambda_i \neq \lambda_j$ introduced in Chapter II is retained as a hypothesis, and with this restriction it is shown that polynomials for which the criterion of Chapter II holds belong to exactly one of four possible classes, the particular class being readily identified by methods which are described. Next the restriction $\lambda_i \neq \lambda_j$ is shown to be a necessary condition for satisfying the criterion developed in Chapter II; that is, satisfaction of that criterion <u>implies</u> that $\lambda_i \neq \lambda_j$ $(i \neq j)$. Hence, in classifying the polynomials which satisfy the criterion of Chapter II, introduction of the requirement $\lambda_i \neq \lambda_j$ $(i \neq j)$ as a separate hypothesis was in fact unnecessary. So the classification developed in the first part of Chapter III is complete in the sense that it includes all polynomials which satisfy the criterion of Chapter II. Finally, two examples are given which illustrate the procedure and the criterion of Chapter II and which also indicate how additional information about the polynomials can be obtained from the differential equation.

Application of the results of Chapters II and III is made in Chapter IV to deduce certain results concerning a class of coupled linear systems. Interest in these systems and a desire for more information about them stems at least in part from the fact that they occur in the study of such physical systems as coupled linear harmonic oscillators.

CHAPTER II

THE RELEVANT DIFFERENTIAL EQUATION

Construction of the Differential Equation

In trying to decide whether the polynomials φ_n generated by (1) are solutions of an equation such as (2), it suffices to consider only equations of the form

$$(\gamma x^2 + \beta x + \alpha)y'' + (\varepsilon x + \delta)y' + \lambda_n y = 0, \quad ' \sim d/dx, \tag{3}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon$ are constants and λ_n is a parameter depending only on n [1]. As will be shown below, $\alpha, \beta, \gamma, \delta, \varepsilon, \lambda_n$ are determined to within a common multiplicative constant -- and consequently (3) is completely determined -- by the first few coefficients in the recurrence relation (1). Note, in particular, that φ_o is a solution of (3) if and only if $\lambda_o = 0$.

<u>Lemma 1</u>. Suppose that for $n = 1,2,3,\ldots,$ $\varphi_n(x)$ is a solution of (3). Then

(i) $\gamma x^2 + \beta x + \alpha = [\lambda_1 - \lambda_2/2]\ x^2$

$$+ [\lambda_1(A_oB_1 + 3A_1B_o) - \lambda_2(A_oB_1 + A_1B_o)]x/2A_oA_1$$

$$+ [\lambda_1(B_oB_1 + A_1B_o^{\,2}/A_o) - \lambda_2(B_oB_1 - C_1)]/2A_oA_1 \ ;$$

(ii) $\varepsilon x + \delta = -\lambda_1(x + B_o/A_o)$;

(iii) $\lambda_n = n(n - 1)(\lambda_2/2) - n(n - 2)\lambda_1, \quad n = 1,2,3,\ldots$.

Proof. The result (ii) is a direct consequence of assuming that $\varphi_1(x)$ is a solution of (3), and (i) follows by use of (ii) from the assumption that $\varphi_2(x)$ is a solution [1]. The expression (iii) for λ_n is obtained by assuming that $\varphi_n(x)$ is a solution of (3) and then differentiating the identity

$$(\gamma x^2 + \beta x + \alpha)\varphi_n''(x) + (\varepsilon x + \delta)\varphi_n'(x) + \lambda_n\varphi_n(x) \equiv 0$$

n times. The result is

$$n(n - 1)\gamma + n\varepsilon + \lambda_n = 0 ,$$

or, by use of (i) and (ii),

$$n(n - 1)(\lambda_1 - \lambda_2/2) + n(-\lambda_1) + \lambda_n = 0,$$

from which (iii) follows.

As seen in Lemma 1, the quantities α, β, γ, δ, ε, λ_n -- and hence the differential equation (3) -- are determined as soon as λ_1 and λ_2 are known. Furthermore, determination of $\alpha, \beta, \gamma, \delta, \varepsilon, \lambda_n$ as prescribed in (i), (ii), (iii) of Lemma 1 insures that $\varphi_1(x)$ and $\varphi_2(x)$ are solutions of (3) no matter what values are assigned to λ_1 and λ_2. There is at this point, of course, no guarantee that the remaining polynomials $\varphi_n(x)$ are solutions; conditions to be derived below will insure that the $\varphi_n(x)$ $(n = 3, 4, 5,\ldots)$ are solutions and will also lead to appropriate values for λ_1 and λ_2.

In the sequel, whenever reference is made to the differential equation (3), it will be understood that $\alpha, \beta, \gamma, \delta, \varepsilon, \lambda_n$ have the values prescribed by Lemma 1.

Conditions under which Each $\varphi_n(x)$ is a Solution of Equation (3)

Necessary and sufficient conditions that the polynomials $\varphi_n(x)$ be solutions of the differential equation (3) are now to be derived. The principal results of this section appear as Theorems I and II.

Definition 1. Given the recurrence relation (1) and the differential equation (3), let

$$\begin{aligned} R_n(x) &= 2A_n(\gamma x^2 + \beta x + \alpha)\varphi_n'(x) \\ &\quad + \left\{(A_n x + B_n)[n\lambda_2 + (1-2n)\lambda_1] + A_n(\varepsilon x + \delta)\right\}\varphi_n(x) \\ &\quad - C_n[(2n-1)\lambda_2 + (4-4n)\lambda_1]\varphi_{n-1}(x), \qquad n = 0,1,2,\ldots , \end{aligned}$$

where by definition $C_o = 0$ and $\varphi_{-1}(x) \equiv 0$.

Lemma 2. The polynomial $\varphi_n(x)$ is a solution of (3) for $n = 1,2,3,\ldots$ if and only if $R_n(x)$ is identically zero in x for $n = 0,1,2,3,\ldots$.

Proof. Let L_n denote the operator

$$(\gamma x^2 + \beta x + \alpha)\, d^2/dx^2 + (\varepsilon x + \delta)d/dx + \lambda_n .$$

A straightforward computation, which makes use of the recurrence relation (1) and part (iii) of Lemma 1, establishes the identity

$$L_{n+1}[\varphi_{n+1}(x)] \equiv (A_n x + B_n)L_n[\varphi_n(x)] - C_n L_{n-1}[\varphi_{n-1}(x)] + R_n(x), \tag{4}$$

$n = 0,1,2,\ldots,$ where, to account for the case $n = 0$, the convention is adopted that $C_o L_{-1}[\varphi_{-1}(x)] \equiv 0$. Now suppose that $R_n(x)$ is identically zero in x for $n = 0,1,2,\ldots$. With $R_o(x) \equiv 0$ and with $\varphi_o(x)$ known to be a solution of (3), the identity (4) for $n = 0$ shows that $\varphi_1(x)$ is also a solution (a fact already known from the way in which (3) was constructed). It then follows by repeated use of (4) that $\varphi_n(x)$ is a solution for every positive integer n. On the other hand, suppose that $\varphi_1, \varphi_2, \varphi_3, \ldots$ are solutions. The conclusion $R_n(x) \equiv 0$ $(n = 0,1,2,\ldots)$ then follows immediately from (4), where for $n = 0$ the fact that φ_o is a solution is also used.

<u>Definition 2</u>. Let $b_n = B_n/A_n$, $n = 0,1,2,\ldots$; $c_n = C_n/A_n A_{n-1}$, $n = 1,2,3,\ldots$. Define $g_1(n)$, $g_2(n)$, $g_3(n)$ as follows for $n = 1,2,3,\ldots$.

$$g_1(n) = [(n+1)b_{n+1} + (-n+1)b_n - b_1 - b_o]\lambda_2$$
$$+ [(-2n-1)b_{n+1} + (2n-3)b_n + b_1 + 3b_o]\lambda_1$$

$$g_2(n) = [(n+1)b_n b_{n+1} - nb_b^2 - b_o b_1 + c_1 - (2n+1)c_{n+1} + (2n-3)c_n]\lambda_2$$
$$+ [(-2n-1)b_n b_{n+1} + (2n-1)b_n^2 + b_o b_1 + b_o^2 + 4nc_{n+1} + (-4n+8)c_n]\lambda_1$$

$$g_3(n) = [-(n+1)b_{n+1} + (2n-1)b_n + (-n+2)b_{n-1}]\lambda_2$$
$$+ [(2n+1)b_{n+1} + (-4n+4)b_n + (2n-5)b_{n-1}]\lambda_1$$

<u>Lemma 3</u>. If $g_1(n) = 0$, $n = 1,2,3,\ldots$, then $g_3(n) = 0$, $n = 1,2,3,\ldots$.

<u>Proof</u>. Since $g_3(1) = -g_1(1)$, $g_1(1) = 0$ implies $g_3(1) = 0$. By addition $g_3(n+1) = g_1(n) - g_1(n+1)$, $n = 1,2,3,\ldots$. Hence $g_1(n) = 0$ for $n = 1,2,3,\ldots$ implies $g_3(n) = 0$ for $n = 2,3,4,\ldots$. Consequently

$g_3(n) = 0$ for every positive integer n.

The necessary and sufficient conditions that the polynomial $\varphi_n(x)$ be a solution of (3) for $n = 1, 2, 3,\ldots,$ which were stated in Lemma 2 in terms of the quantities $R_n(x)$, can now be stated in terms of A_n, B_n, C_n alone, or equivalently in terms of b_n and c_n alone.

Theorem I. The polynomial $\varphi_n(x)$ is a solution of (3) for $n = 1,2,3,\ldots$ if and only if $g_1(n) = g_2(n) = 0$ for every positive integer n.

Proof. As in Lemma 2, the proof hinges on an identity -- namely,

$$R_{n+1}(x) \equiv A_{n+1}(x + b_n)R_n(x) - A_{n+1}C_nR_{n-1}(x)/A_{n-1} + g_3(n)\varphi_{n-1}(x) \qquad (5)$$
$$+ g_2(n)A_nA_{n+1}\varphi_n(x) + g_1(n)A_nA_{n+1}x\varphi_n(x)\,, \quad n = 1,2,3,\ldots\,.$$

The computation needed to establish this identity is again straightforward, but lengthy; use is made of Definition 1, the recurrence relation (1), and part (iii) of Lemma 1.

To prove the sufficiency, suppose that $g_1(n) = g_2(n) = 0$ for every positive integer n. Then by Lemma 3, $g_3(n) = 0$ for $n = 1,2,3,\ldots,$ and the identity (5) reduces to

$$R_{n+1}(x) = A_{n+1}(x+b_n)R_n(x) - A_{n+1}C_nR_{n-1}(x)/A_{n-1}, \quad n = 1,2,3,\ldots\,. \qquad (6)$$

From the definition of $R_n(x)$ it follows by use of Lemma 1 that $R_o(x)$ and $R_1(x)$ are identically zero in x (this is a consequence of the fact that (3) was so constructed that $\varphi_1(x)$ and $\varphi_2(x)$ are solutions regardless of the values assigned to λ_1 and λ_2). Hence, by repeated use of (6), $R_n(x)$ is identically zero in x, $n = 2,3,4,\ldots;$ and so by

Lemma 2, $\varphi_n(x)$ is a solution of (3) for every positive integer n.

Conversely, suppose that $\varphi_n(x)$ is a solution of (3), $n = 1,2,3,\ldots$. Then by Lemma 2, $R_n(x) \equiv 0$, $n = 0,1,2,\ldots$, and the identity (5) becomes

$$g_3(n)\varphi_{n-1}(x) + g_2(n)A_nA_{n+1}\varphi_n(x) + g_1(n)A_nA_{n+1}x\,\varphi_n(x) \equiv 0,$$
$$n = 1,2,3,\ldots .$$

But the polynomials $\varphi_{n-1}(x)$, $\varphi_n(x)$, $x\varphi_n(x)$ are linearly independent on any finite interval of the real line; hence $g_1(n) = g_2(n) = g_3(n) = 0$ for every positive integer n.

Thus far nothing has been said about appropriate values for λ_1 and λ_2 (except to note that whatever values λ_1 and λ_2 may have, $\varphi_1(x)$ and $\varphi_2(x)$ are solutions of (3)). If specific values are assigned to λ_1 and λ_2, then Theorem I states conditions under which the remaining $\varphi_n(x)$, $n = \cancel{2},3,4,\ldots$, will also be solutions. In particular, if λ_1 and λ_2 are not both to be zero (if they were, (3) would be trivial, as can be seen from Lemma 1) and if the conditions of Theorem I are to be met for $n = 1$, the determinant Δ of coefficients of λ_1 and λ_2 in the system

$$g_1(1) = (2b_2 - b_1 - b_o)\lambda_2 + (-3b_2 + 3b_o)\lambda_1 = 0 \tag{7}$$

$$\begin{aligned} g_2(1) = {} & (2b_1b_2 - b_1^2 - b_ob_1 - 3c_2)\lambda_2 \\ & + (-3b_1b_2 + b_1^2 + b_ob_1 + b_o^2 + 4c_1 + 4c_2)\lambda_1 = 0 \end{aligned}$$

must be zero; that is,

$$\Delta \equiv (2b_2 - b_1 - b_o)[(b_1 - b_o)^2 + 4(c_1 + c_2)] + 9c_2(b_o - b_2) = 0 . \qquad (8)$$

Thus, if and only if $\Delta = 0$, it is possible to assign an arbitrary non-zero value to one or the other of the quantities λ_1 or λ_2 and then to compute the corresponding value of λ_2 or λ_1 from one of the equations (7). In fact, the non-zero value may be assigned to λ_1, because (since $c_2 \neq 0$) equations (7) imply that if $\lambda_1 = 0$, then also $\lambda_2 = 0$. Accordingly, in the remainder of this paper it will be assumed that $\Delta = 0$ and that $\lambda_1 = 1$, which in turn imply that $\lambda_2 = [(b_1 - b_o)^2 + 4(c_1 + c_2)]/3c_2$. With these values of λ_1 and λ_2 inserted in the expressions for $g_1(n)$ and $g_2(n)$, Theorem I provides a necessary and sufficient condition that $\varphi_n(x)$ be a solution of the non-trivial differential equation (3) for $n = 1,2,3,\ldots$.

A Modification of Theorem I when $\lambda_i \neq \lambda_j$

With $\lambda_1 = 1$ it follows from (iii) in Lemma 1 that $\lambda_i = \lambda_j$ for some i and j $(i \neq j,\ i + j \geq 2)$ if and only if λ_2 has the value $2(i + j - 2)/(i + j - 1)$. Consequently, the requirement that $\lambda_2 = [(b_1 - b_o)^2 + 4(c_1 + c_2)]/3c_2$ be different from $2(m - 2)/(m - 1)$ for every integer $m \geq 2$, together with the fact that $\lambda_o = 0$, insures that $\lambda_i \neq \lambda_j$ $(i \neq j;\ i,\ j = 0, 1, 2,\ldots)$.

Theorem I and the comments subsequent to it can now be summarized as a single theorem.

Theorem II. $\varphi_n(x)$, $n = 1,2,3,\ldots,$ is a solution of the non-trivial differential equation (3), in which $\lambda_i \neq \lambda_j$ $(i \neq j;\ i,j = 0,1,2,\ldots)$, if and only if

(i) $\Delta = 0$;

(ii) $[(b_o - b_1)^2 + 4(c_1 + c_2)]/3c_2$ is different from $2(m - 2)/(m - 1)$ for every integer $m \geq 2$;

(iii) $g_1(n) = g_2(n) = 0$, $n = 1,2,3,\ldots,$ where $g_1(n)$, $g_2(n)$ are evaluated with $\lambda_1 = 1$ and $\lambda_2 = [(b_o - b_1)^2 + 4(c_1 + c_2)]/3c_2$.

Some Remarks on Simplifying Condition (iii) in Theorem II

With the understanding that $\Delta = 0$ and that λ_1 and λ_2 have the values prescribed in part (iii) of Theorem II, the conditions $g_1(n) = 0$, $g_2(n) = 0$, $n = 1,2,3,\ldots,$ can be regarded as a pair of linear first-order difference equations:

$$\begin{aligned} &[(n + 1)\lambda_2 - (2n + 1)]b_{n+1} - [(n - 1)\lambda_2 - (2n - 3)]b_n \\ &\qquad = (b_o + b_1)\lambda_2 - b_1 - 3b_o \; ; \\ &[(2n + 1)\lambda_2 - 4n]c_{n+1} - [(2n - 3)\lambda_2 - (4n - 8)]c_n \\ &\qquad = [(n + 1)b_n b_{n+1} - nb_n^2 - b_o b_1 + c_1]\lambda_2 \\ &\qquad\quad + (-2n - 1)b_n b_{n+1} + (2n - 1)b_n^2 + b_o b_1 + b_o^2 , \\ &\qquad\qquad n = 1,2,3,\ldots . \end{aligned} \tag{9}$$

The requirement that λ_2 be different from $2(m - 2)/(m - 1)$ for every integer $m \geq 2$ has an interesting consequence here: it substantially simplifies condition (iii) in Theorem II. For suppose that $m - 1$ is even -- say $m - 1 = 2(n + 1)$. Then $\lambda_2 \neq (2n + 1)/(n + 1)$. And if, on the other hand, $m - 1$ is odd -- say $m - 1 = 2n + 1$ -- then $\lambda_2 \neq 4n/(2n + 1)$. It follows that the difference equations (9) have no singular points and consequently, in particular, that the solution for b_n is

$$b_n = \frac{[(b_o + b_1)\lambda_2 - (b_1 + 3b_o)][(n-1)\lambda_2 - (2n-4)]n - 2b_o(\lambda_2 - 3)}{2[n\lambda_2 - (2n-1)][(n-1)\lambda_2 - (2n-3)]} ,$$

$$n = 1,2,3,\ldots . \qquad (10)$$

Thus the requirement $g_1(n) = 0$ is equivalent to the requirement that b_n have the value (10), which is of the form

$$b_n = \frac{r_o n^2 + r_1 n + r_2}{s_o n^2 + s_1 n + s_2} ,$$

where r_j and s_j $(j = 0,1,2)$ are independent of n. The equation for c_n is less tractable because of the relatively complicated nonhomogeneous term, but in the special case $\lambda_2 = 2$ (whose importance will be indicated later) the requirements $g_1(n) = g_2(n) = 0$ are equivalent to

$$b_n = b_o + (b_1 - b_o)n , \qquad (11)$$

$$c_n = [n(b_1 - b_o)]^2/4 + n[c_1 - (b_1 - b_o)^2/4] ,$$

$$n = 1,2,3,\ldots .$$

If $\lambda_2 \neq 2$, the special case $b_o = b_1$ is of interest. For then (10) shows that $b_n = b_o$, $n = 1,2,3,\ldots$, and consequently the difference equation for c_n simplifies to

$$[(2n+1)\lambda_2 - 4n]c_{n+1} - [(2n-3)\lambda_2 - (4n-8)]c_n = c_1\lambda_2 ,$$

$$n = 1,2,3,\ldots,$$

whose solution is

$$c_n = \frac{c_1\lambda_2[(n - 2)\lambda_2 - 2(n - 3)]\, n}{[(2n - 1)\lambda_2 - 4(n - 1)][(2n - 3)\lambda_2 - 4(n - 2)]}\,,$$

$$n = 1,2,3,\ldots\,.$$

If additionally $c_1 = c_2$ (in which case $\lambda_2 = 8/3$), then the above formula reduces to $c_1 = c_n$ for every positive integer n. Thus, if $b_o = b_1$ and $c_1 = c_2$, the requirements $g_1(n) = g_2(n) = 0$ reduce to $b_n = b_o$, $c_n = c_1$, $n = 1,2,3,\ldots$.

CHAPTER III

CLASSIFICATION OF THE POLYNOMIALS GENERATED

Classification under the Assumption that $\lambda_i \neq \lambda_j$

If the coefficients in the recurrence relation (1) satisfy the hypotheses of Theorem II, the polynomials generated by (1) satisfy (3). The nature of the polynomials then hinges on the coefficients of y'' and y' in (3) in a way now to be explained.

Consider $\gamma x^2 + \beta x + \alpha$, the coefficient of y''. Note first that this coefficient cannot be identically zero. For if it were, then from (i) of Lemma 1 it would follow that $\lambda_2 = 2$ (since λ_1 has been chosen to be one), $A_oB_1 = A_1B_o$, and $A_1B_o^2/A_o - B_oB_1 + 2C_1 = 0$. But the last two equalities imply that $2C_1 = 0$, a contradiction of the hypothesis that C_n is different from zero for every positive n. There then remain four possibilities for $\gamma x^2 + \beta x + \alpha$, each of which will be discussed in turn:

(i) it is a non-zero constant;

(ii) it is linear in x;

(iii) it has a double zero;

(iv) it has two distinct zeros.

Suppose first that the coefficient is a non-zero constant. Then $\gamma = \beta = 0$ and (3) has the form

$$c_1 y'' - (x + b_o)y' + ny = 0, \quad ' \sim d/dx \,, \tag{12}$$

which becomes

$$y'' - 2ty' + 2ny = 0, \quad ' \sim d/dt \,, \tag{13}$$

under the change of variable $x = \sqrt{2c_1}\,t - b_o$. The only polynomial solutions of this equation are the Hermite polynomials [11; pp. 104, 105].

If $\gamma x^2 + \beta x + \alpha$ is linear in x, then $\gamma = 0$, $\beta \neq 0$, and (3) has the form

$$[(b_o - b_1)x + (b_o^2 - b_o b_1 + 2c_1)]y'' - 2(x + b_o)y' + 2ny = 0, \quad ' \sim d/dx. \tag{14}$$

With $x = (b_o - b_1)t/2 - b_o - 2c_1/(b_o - b_1)$, (14) becomes

$$ty'' + [4c_1/(b_o - b_1)^2 - t]y' + ny = 0, \quad ' \sim d/dt \,,$$

which can be written

$$ty'' + (a + 1 - t)y' + ny = 0, \quad ' \sim d/dt \,, \tag{15}$$

by setting $4c_1/(b_o - b_1)^2 = a + 1$. Here $a \neq -1$ since by hypothesis $c_1 \neq 0$. In fact, the number a cannot be any other negative integer; for if so (say $a = -K$, $K \neq 1$, K a positive integer), then $(b_1 - b_o)^2/4 = c_1/(1 - K)$. But $\gamma = 0$ implies $\lambda_2 = 2$, which in turn implies by use of (11) that $c_n = nc_1(n - K)/(1 - K)$. This formula for c_n in terms of c_1 implies, when $n = K$, that $c_K = 0$, a contradiction. With $\underline{a}$ not equal to any negative integer, the only polynomial solutions of (15) are the Laguerre polynomials [11; pp. 99, 101].

Next suppose that $\gamma x^2 + \beta x + \alpha$ has a double zero, say x^*. Then the discriminant of $\gamma x^2 + \beta x + \alpha$ is zero -- that is, $(\lambda_2 - 1)^2(b_o - b_1)^2 + 4\lambda_2(\lambda_2 - 2)c_1 = 0$ -- and (3) becomes

$$\begin{aligned}(2 - \lambda_2)(x - x^*)^2 y'' - 2(x + b_o)y' \\ + 2[n(n - 1)\lambda_2/2 - n(n - 2)]y = 0 \,,\end{aligned} \tag{16}$$

where $\lambda_2 \neq 2$ and $x^* = [\lambda_2(b_o + b_1) - (3b_o + b_1)]/2(2 - \lambda_2)$. Here the change of variable $x = t + x^*$ transforms (16) into

$$t^2y'' + [2t/(\lambda_2 - 2) + (b_o - b_1)(\lambda_2 - 1)/(\lambda_2 - 2)^2]y' - n[n - 1 + 2/(\lambda_2 - 2)]y = 0, \quad ' \sim d/dt . \tag{17}$$

The coefficient of t in (17) cannot be a negative integer, for if $2/(\lambda_2 - 2) = - N$ (N a positive integer), then $\lambda_2 = 2(N-1)/N$, which contradicts (ii) in Theorem II. Note also that $(b_o - b_1)(\lambda_2 - 1) \neq 0$; for if it were, then the discriminant would reduce to $4\lambda_2(\lambda_2 - 2)c_1$, which cannot be zero since $\lambda_2 \neq 0$, $\lambda_2 \neq 2$, $c_1 \neq 0$. Consequently the only polynomial solutions of (17) are the generalized Bessel polynomials, first studied extensively by Krall and Frink [7].

Finally suppose that $\gamma x^2 + \beta x + \alpha$ has distinct zeros x_1 and x_2. Equation (3) here has the form

$$(1 - \lambda_2/2)(x - x_1)(x - x_2)y'' - (x + b_o)y' + [n(n - 1)\lambda_2/2 - n(n - 2)]y = 0 , \tag{18}$$

where again $\lambda_2 \neq 2$, and x_1, x_2 can be computed from the quadratic formula. The change of variable $x = x_1 + (x_2 - x_1)t$ transforms (18) into

$$t(1 - t)y'' + [a + 1 - (a + b + 2)t]y' + n(n + a + b + 1)y = 0, \quad ' \sim d/dt , \tag{19}$$

where $a + b + 2 = 2/(\lambda_2 - 2)$ and $a + 1 = 2(b_o + x_1)/(\lambda_2 - 2)(x_1 - x_2)$. Since $a + b + 1 = -1 + 2/(\lambda_2 - 2)$ cannot be a negative integer (for the same reason that the coefficient of t in (17) could not be a negative integer), it follows that the only polynomial solutions of (19) are the Jacobi polynomials [11; pp. 60-65].

The discussion just completed has shown that under the hypotheses of Theorem II the polynomials generated by (1) are -- apart from the indicated changes of variable and multiplicative factors which may depend on n but not on x -- either the Hermite, Laguerre, generalized Bessel or Jacobi polynomials, the particular class depending on the nature of $\gamma x^2 + \beta x + \alpha$. These results may be summarized briefly in the following way.

Case I. $\lambda_2 = 2$.

a) If $b_o = b_1$, the polynomials are the Hermite polynomials.

b) If $b_o \neq b_1$, the polynomials are the Laguerre polynomials.

Case II. $\lambda_2 \neq 2$.

a) If the discriminant of $\gamma x^2 + \beta x + \alpha$ is zero, the polynomials are the generalized Bessel polynomials.

b) If the discriminant is not zero, the polynomials are the Jacobi polynomials.

Discussion of the Restriction $\lambda_i \neq \lambda_j$

In this section it will be shown that if the hypotheses of Theorem I are satisfied and $\Delta = 0$, then in the resulting differential equation it is impossible for λ_i to be equal to λ_j if $i \neq j$. Consequently in Theorem II and in the classification developed in the preceding section it is unnecessary to <u>hypothesize</u> this inequality. It is correct simply to state that polynomials generated by (1) which are also solutions of (3) can belong only to one of the four classes already described; for the possibility, which has been left open up to this point, of generating additional classes of polynomials when $\lambda_i = \lambda_j$ $(i \neq j)$ does not in fact exist.

As has been previously noted, the condition $\lambda_i = \lambda_j$ for some i, j, $(i \neq j)$, is equivalent to the condition $\lambda_2 = 2(i+j-2)/(i+j-1)$. So, to obtain the results summarized in the preceding paragraph, it will be assumed that λ_2 has this special value. Then, from the particular form which the identities $g_1(n) = 0$, $g_2(n) = 0$ take under this assumption, certain contradictions will be extracted and consequently the special value will be excluded. The two cases $i + j$ odd, $i + j$ even, will be considered separately.

<u>Case I</u>. Suppose $i + j$ is odd, say $i + j = 2K + 1$ for some integer $K \geq 1$ (since $\lambda_o = 0$, $\lambda_1 = 1$, the possibility $i + j = 1$ need not be considered). Then λ_2 equals $(2K - 1)/K$ and the condition $g_1(n) = 0$, $n \geq 1$, reduces to

$$(K-n-1)b_{n+1} - (K-n+1)b_n = (K-1)b_1 - (K+1)b_o, \quad n \geq 1 . \tag{20}$$

<u>Lemma 4</u>. The difference equation (20) is satisfied if and only if $b_n = b_o$ $(n = 0,1,\ldots,K-1, K+2, K+3,\ldots)$ and $b_{K+1} = 2b_o - b_K$, where b_K is arbitrary.

<u>Proof</u>: If $K = 1$, then (20) becomes

$$nb_{n+1} + (2 - n)b_n = 2b_o ,$$

from which it follows that b_1 is arbitrary, $b_2 = 2b_o - b_1$, $b_3 = b_o$, and then by induction that $b_n = b_o$, $n \geq 3$. Now suppose that $K > 1$. Then for $n = K - 2$ (20) yields

$$b_{K-1} - 3b_{K-2} = (K - 1)b_1 - (K + 1)b_o ,$$

and for $n = K - 1$ (20) yields (for arbitrary b_K)

$$-2b_{K-1} = (K - 1)b_1 - (K + 1)b_o .$$

These two equations imply that

$$b_{K-2} = b_{K-1} = [(K + 1)b_o - (K - 1)b_1]/2 .$$

Suppose for some integer n, $1 \leq n \leq K - 2$, that

$$b_{n+1} = [(K + 1)b_o - (K - 1)b_1]/2 .$$

Equation (20) now implies that

$$(K - n - 1)[(K + 1)b_o - (K - 1)b_1] - 2(K - n + 1)b_n$$
$$= 2[(K - 1)b_1 - (K + 1)b_o] ,$$

or that $b_n = b_{K-1} = [(K + 1)b_o - (K - 1)b_1]/2$ for $n = 1,\ldots,K - 2$. But a direct computation from (20) shows that if $b_2 = b_1$, then $b_1 = b_o$ and consequently $b_n = b_o$, $n = 0,1,\ldots,K - 1$, while b_K is arbitrary. Now for $n = K$, (20) yields $b_{K+1} = 2b_o - b_K$ and for $n = K = 1$, $b_{K+2} = b_o$. It then follows by induction that $b_n = b_o$,

$n \geq K + 2$.

Thus Lemma 4 shows that if $\lambda_2 = (2K - 1)/K$, then the condition $g_1(n) = 0$ will be satisfied for every positive integer n if and only if $b_n = b_o$ $(n = 0,\ldots,K-1, K+2, K+3,\ldots)$ and $b_{K+1} = 2b_o - b_K$, where b_K is arbitrary.

Now consider the condition $g_2(n) = 0$, $n \geq 1$. The value $\lambda_2 = (2K - 1)/K$ does not introduce a singular point into this equation, and if b_K is chosen equal to b_o then $b_n = b_o$ for <u>every</u> n, so that the condition on c_n is exactly that prescribed on page 13. With $\lambda_2 = (2K - 1)/K$ this condition is

$$c_n = \frac{c_1(2K - 1)(2K + 2 - n)n}{(2K + 1 - 2n)(2K + 3 - 2n)}\,, \tag{21}$$

$n = 1,2,3,\ldots$. In fact the result (21) is valid for $n \geq K + 2$ even if b_K is <u>not</u> chosen equal to b_o. To see this, suppose first that $K > 2$ and note that in any case (21) is valid for $1 \leq n \leq K - 2$, since $b_n = b_o$ for $1 \leq n \leq K - 1$. In particular $c_{K-2} = c_1(2K-1)(K-2)(K+4)/35$. Now with b_K arbitrary, $g_2(K - 2) = 0$ is equivalent to

$$[(2K-1) - (2K-4)]c_{K-1} - [(2K+3) - (2K-4)]c_{K-2} = c_1(2K-1)\ ,$$

from which it follows by use of the known value for c_{K-2} that $c_{K-1} = c_1(2K - 1)(K + 3)(K - 1)/15$. In turn the condition $g_2(K-1) = 0$ is equivalent to

$$[(2K-1) - (2K-2)]c_K - [(2K+3) - (2K-2)]c_{K-1} = c_1(2K-1)\ ,$$

whence $c_K = c_1(2K - 1)(K + 2)K/3$. This condition is followed by $g_2(K) = 0$ which is equivalent to

$$c_{K+1} + 3c_K = -(b_K - b_o)^2 - c_1(2K - 1)$$

or

$$c_{K+1} = -(b_K - b_o)^2 - 2(2K - 1)(K + 1)^2 c_1 .$$

Finally, $g_2(K + 1) = 0$ is equivalent to

$$[(2K - 1) - (2K + 2)]c_{K+2} - [(2K + 3) - (2K + 2)]c_{K+1} = c_1(2K - 1) + (b_K - b_o)^2 ,$$

and with the previously computed value of c_{K+1} the value of c_{K+2} is found to be

$$c_{K+2} = c_1(2K - 1)(K + 2)K/3 .$$

But this result agrees with (21) when $n = K + 2$; and since $b_n = b_o$ for $n \geq K + 2$, it follows that (21) is valid for $n \geq K + 2$. A similar argument with $K = 1$ shows that (21) is valid for $n \geq 3$, while with $K = 2$ it is valid for $n \geq 4$. These observations lead to the following theorem.

<u>Theorem III</u>. Suppose a sequence of polynomials generated by (1) satisfies the hypotheses of Theorem I with $\Delta = 0$. Then $\lambda_2 \neq (2K - 1)/K$.

<u>Proof</u>: If $\lambda_2 = (2K - 1)/K$, then from (21) $c_{2K+2} = 0$, which contradicts the requirement $A_n C_n \neq 0$, since $c_n = C_n/A_n A_{n-1}$.

Case II. Suppose $i + j$ is even, say $i + j = 2K + 2$, for some integer $K \geq 0$. Then $\lambda_2 = 4K/(2K + 1)$, and the condition $g_1(n) = 0$ is equivalent to

$$(2K - 1 - 2n)b_{n+1} - (2K - 3 - 2n)b_n = (2K - 1)b_o - (2K + 3)b_1 , \tag{22}$$

$n \geq 1$. Since there is no singular point in this equation, the solution (10) is valid and

$$b_n = \frac{[(2K+3)b_o - (2K - 1)b_1](n - 2K - 1)(n - 1) + b_1(4K^2 - 1)}{(2n - 2K - 1)(2n - 2K - 3)}, \tag{23}$$

$n \geq 1$. The condition $g_2(n) = 0$ is equivalent to

$$4[(K - n)c_{n+1} - (K - 2 - n)c_n] = (2K - 1 - 2n)b_n b_{n+1}$$

$$- (2K + 1 - 2n)b_n^2 - (2K - 1)b_o b_1 + (2K + 1)b_1^2 + 4Kc_1 ,$$

$n \geq 1$, which may be reduced by application of (22) and (23) to

$$(K - n)c_{n+1} - (K + 2 - n)c_n = Kc_1 \tag{24}$$

$$+ \frac{(b_1 - b_o)^2(2K - 1)^2[2n^2 - 4(K+1)n + (2K+3)(2K+1)](2K+2-n)n}{4[2n - (2K + 1)]^2[2n - (2K + 3)]^2}$$

for $n \geq 1$. Equation (24) furnishes the basis for the proof of the following theorem.

Theorem IV. Suppose the hypotheses of Theorem I are satisfied and that $\Delta = 0$. Then $\lambda_2 \neq 4K/(2K + 1)$.

Proof: If $\lambda_2 = 4K/(2K + 1)$, then the requirement $g_2(n) = 0$ is equivalent to (24) above. If (24) has no solution, this requirement cannot be met, contrary to hypothesis. Suppose then that (24) has a solution. The right side of this equation is symmetric about the point $n = K + 1$; that is, the right side has the same value for $n = K + 1 + k$ as for $n = K + 1 - k$, where k is any integer satisfying $0 \leq k \leq K$. Consequently the left side of (24) has the same values for $n = K + 1 + k$ and $n = K + 1 - k$; that is,

$$(k-1)c_{K+2-k} - (k+1)c_{K+1-k} = -(k+1)c_{K+2+k} + (k-1)c_{K+1+k}, \qquad (25)$$

$$0 \leq k \leq K .$$

For $k = 1$, (25) simplifies to $c_K = c_{K+3}$, and for $k = 2$ it simplifies to

$$c_K - 3c_{K-1} = -3c_{K+4} + c_{K+3} .$$

But since $c_K = c_{K+3}$, it follows that $c_{K-1} = c_{K+4}$ and from this point it is easy to show by induction that $c_{K-2} = c_{K+5}, \ldots, c_1 = c_{2K+2}$. With $n = 2K + 2$, equation (25) reduces to

$$-(K + 2)c_{2K+3} + Kc_{2K+2} = Kc_1 ,$$

and since $c_1 = c_{2K+2}$, it follows that $c_{2K+3} = 0$. As in Theorem III, this result is a contradiction because $c_n \neq 0$, $n \geq 1$.

Illustrations of the Procedure for Classifying the Polynomials Generated

As an example, consider the polynomials generated by

$$\varphi_o = 1 \tag{26}$$
$$\varphi_1 = x$$
$$\varphi_{n+1} = 2x\varphi_n + \varphi_n ,$$

$n = 1,2,3,\ldots$. Here $b_n = 0$ for every n; so the conditions $\Delta = 0$, $g_1(n) = 0$ $(n = 1,2,3,\ldots)$ in Theorem II are fulfilled. Also, $c_1 = -1/2$, $c_n = -1/4$ $(n = 2,3,4,\ldots)$; hence $\lambda_2 = 4$ (which implies that condition (ii) is satisfied), and $g_2(n) = 0$ $(n = 1,2,3,\ldots)$. Thus all the hypotheses of Theorem II are satisfied, and the polynomials generated by (26) are solutions of the differential equation (3), which for this example is

$$(-x^2 - 1)y'' - xy' + n^2y = 0, \quad ' \sim d/dx .$$

Under the change of variable $x = it$ $(i = \sqrt{-1})$ this equation becomes

$$(1 - t^2)y'' - ty' + n^2y = 0, \quad ' \sim d/dt .$$

Consequently the polynomials are, except for multiplicative factors depending on n but not on x, the Tchebycheff polynomials of the first kind with argument $-ix$, that is, $T_n(-ix)$. From this result it follows, as a matter of some interest, that all the zeros of each polynomial lie on the imaginary axis between $-i$ and i in the x-plane.

As another example, consider the recurrence relation

$$\varphi_o = 1 \tag{27}$$
$$\varphi_1 = x + 1$$
$$\varphi_{n+1} = (x+2)\varphi_n - \varphi_{n-1} ,$$

$n = 1,2,3,\ldots$. Here $b_0 = 1$, $b_n = 2$ $(n = 1,2,3,\ldots)$, $c_n = 1$ $(n = 1,2,3,\ldots)$, $\lambda_2 = 3$; and it is readily verified that the hypotheses of Theorem II are satisfied. The polynomials generated are of the Jacobi type, and the differential equation is

$$-x(x + 4)y'' - 2(x + 1)y' + n(n + 1)y = 0, \qquad ' \sim d/dx ,$$

which, after introduction of a suitable integrating factor, becomes

$$\frac{d}{dx}\left[(-x)^{1/2} (x + 4)^{3/2} \frac{dy}{dx}\right] + n(n+1)(-x)^{-1/2}(x+4)^{1/2} y = 0.$$

From this latter form it can be shown in the usual way that the polynomials are orthogonal; here the interval of orthogonality is $(-4,0)$, and the weight function is $(-x)^{-1/2}(x + 4)^{1/2}$. That all the zeros of each of the polynomials lie in the interval $(-4,0)$ follows from a well-known property of orthogonal polynomials.

CHAPTER IV

APPLICATIONS

In this chapter the conclusions reached in the previous chapters are used to deduce some results concerning a class of coupled linear systems of which the coupled harmonic oscillators shown in Figure 1 may be regarded as a prototype.

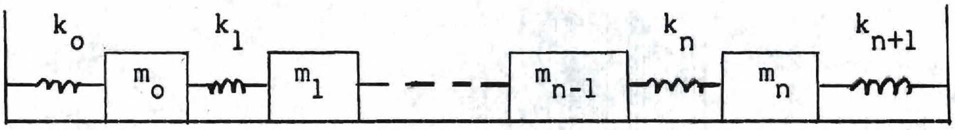

Figure 1. A System of Coupled Harmonic Oscillators.

The study is motivated by the following observations. If one mass m_o is coupled to two linear weightless springs having spring constants k_o, k_1, -- i.e., $n = 0$ in Figure 1 -- the equation of motion in the absence of friction is

$$(m_o D^2 + k_o + k_1)x_o = 0 ,$$

where the operator D indicates differentiation with respect to time and x_o is the displacement of the mass from equilibrium. If a second mass m_1 and third spring with spring constant k_2 are added in such a way that the resulting system corresponds to $n = 1$ in Figure 1, the equations of motion are

$$(m_o D^2 + k_o + k_1)x_o - k_1 x_1 = 0$$

$$- k_1 x_o + (m_1 D^2 + k_1 + k_2)x_1 = 0 .$$

Arbitrarily large systems can be constructed in this manner by adding additional masses and springs, and the equations of motion for n masses and $n + 1$ springs are

$$\begin{aligned}
(m_o D^2 + k_o + k_1)x_o - k_1 x_1 &= 0 \\
- k_1 x_o + (m_1 D^2 + k_1 + k_2)x_1 - k_2 x_2 &= 0 \\
&\vdots \\
- k_i x_{i-1} + (m_i D^2 + k_i + k_{i+1})x_i - k_{i+1} x_{i+1} &= 0 \\
&\vdots \\
- k_{n-1} x_{n-2} + (m_{n-1} D^2 + k_{n-1} + k_n)x_{n-1} &= 0 .
\end{aligned}$$

The characteristic polynomials for these systems are obtained by assuming solutions of the form $x_j = A_j e^{i\omega t}$; they are

$$\begin{aligned}
\varphi_1 &= -m_o \omega^2 + k_o + k_1 \\
\varphi_2 &= (-m_1 \omega^2 + k_1 + k_2)\varphi_1 - k_1^2 \\
\varphi_3 &= (-m_2 \omega^2 + k_2 + k_3)\varphi_2 - k_2^2 \varphi_1 \\
&\vdots \\
\varphi_n &= (-m_{n-1} \omega^2 + k_{n-1} + k_n)\varphi_{n-1} - k_{n-1}^2 \varphi_{n-2} .
\end{aligned}$$

Thus if φ_o is defined to be 1 and if the characteristic polynomials for the systems of orders $n-1$, n, $n+1$ are viewed as polynomials in ω^2, they are related by a three-term recursion formula of the type studied in the previous chapters, with

$$b_n = -(k_n + k_{n+1})/m_n, \quad n = 0,1,2,\ldots, \tag{28}$$

and

$$c_n = k_n^2/m_n m_{n-1}, \quad n = 1,2,3,\ldots . \tag{29}$$

Note that each b_n is negative and each c_n is positive, since the spring constants and masses are positive with the possible exception of k_o, which may be zero.

An interesting question is prompted by these observations: Starting with a given mass m_o and two springs with given spring constants k_o and k_1 as previously indicated, is it possible, by successively adding a mass m_j and a spring with spring constant k_{j+1} $(j = 1,2,3,\ldots)$, to construct a system such that the resulting expressions for b_n and c_n satisfy the hypotheses of Theorem II? This question and some of its ramifications are investigated in detail in this chapter.*

To begin with, it is necessary to indicate how the additional spring constants and masses are to be determined. Notice first that the hypotheses of Theorem II impose no essential restrictions on b_o, b_1, c_1, since with an arbitrary b_o, b_1, and c_1 it is always possible to determine a positive c_2 and a negative b_2 so that $\Delta = 0$ and so that λ_2 is not of the

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*It is important to note at this juncture a related question which is not being asked: Given a positive integer N, is it possible to choose $k_o, m_o, k_1, m_1, k_2, \ldots, m_N, k_{N+1}$ in such a way that hypotheses (i) and (ii) of Theorem II are satisfied and $g_1(n) = 0 = g_2(n)$ for $n \leq N$?

form $2(m - 2)/(m - 1)$, where m is a positive integer not less than two. Hence b_o, b_1, c_1 may be regarded as arbitrary; and consequently, no matter how m_o, k_o and k_1 have been given, m_1 and k_2 may be chosen arbitrarily. Once b_2 and c_2 have been adjusted to make $\Delta = 0$, the value of λ_2 can be computed, and m_2 is determined by (29). If the computed value of m_2 is positive it is inserted in (28) along with the value of b_2, and k_3 is computed. If $k_3 \leq 0$, the proposed construction is physically unrealizable. If $k_3 > 0$, the computation is continued as follows: The value of b_3 is computed from (10), and c_3 is computed from the second of equations (9) with $n = 2$. The values for m_3 and k_4 are now computed from (28) and (29); if both prove to be positive the process is continued.

By using equations (10), (9), (28), and (29), it is possible to compute alternately the successive values of the m_n's and k_n's for arbitrarily large values of n, and these computed quantities may be given the desired physical interpretation as long as they are positive. On the other hand, if for some n the computed k_n or m_n is non-positive, then it is impossible to add a spring and mass in such a way that the hypotheses of Theorem II are satisfied. Which of these alternatives will actually be observed is determined by the values of the parameters m_o, m_1, m_2, k_o, k_1, k_2, k_3. Since these values also determine the type of polynomial generated by satisfying the hypotheses of Theorem II, each of the polynomial classes obtained in Chapter III will be considered separately.

The first two cases to be investigated are those for which $\lambda_2 = 2$. If the hypotheses of Theorem II are satisfied and $\lambda_2 = 2$, then from (11) either $b_n = 0$ and $c_n = nc_1$ $(n = 1,2,3,\ldots)$, in which case the polynomials are the Hermite polynomials, or

$$b_n = b_o + (b_1 - b_o)n, \quad b_o \neq b_1 ,$$

$$c_n = [n(b_1 - b_o)]^2/4 + n[c_1 - (b_1 - b_o)^2/4]$$

$(n = 1,2,3,\ldots)$, in which case the polynomials are the Laguerre polynomials.

In terms of spring constants k_n and masses m_n, the first of these possibilities is equivalent to

$$(k_n + k_{n+1})/m_n = (k_o + k_1)/m_o, \quad n = 1,2,3,\ldots, \tag{30}$$

$$k_n^2/m_n m_{n-1} = nk_1^2/m_1 m_o, \quad n = 1,2,3,\ldots, \tag{31}$$

as may be seen from (28) and (29). By using (30) and (31), the following theorem can be proved.

<u>Theorem V</u>. For a given set of values m_o, m_1, k_o, k_1, k_2, it is impossible, by adding springs and masses as shown in Figure 1, to construct a system of harmonic oscillators such that (30) and (31) hold for every positive integer n. Thus for a given set of values m_o, m_1, k_o, k_1, k_2, it is impossible to construct a system such that for every positive integer n the characteristic polynomial is -- apart from a linear change of variable -- a Hermite polynomial in ω^2.

<u>Proof</u>. Suppose otherwise. Then (30), (31) hold for n arbitrarily large. From (31) it follows that

$$(k_n/m_n)^2 = nm_{n-1}k_1^2/m_o m_1 m_n \ ,$$

$$(k_{n+1}/m_n)^2 = (n + 1)m_{n+1}k_1^2/m_o m_1 m_n \ ;$$

and when these expressions are substituted into (30) the result is (after division by $\sqrt{n}$)

$$(k_o + k_1)/m_o\sqrt{n} = [(m_n/m_{n-1})^{-1/2} + (1 + 1/n)^{1/2}(m_{n+1}/m_n)^{1/2}]k_1/\sqrt{m_o m_1} \ . \tag{32}$$

Suppose $\lim_{n\to\infty} m_{n+1}/m_n = L^2 > 0$. Then when the limits of both sides of (32) are taken, the result is $0 = (1/L + L)k_1/\sqrt{m_o m_1}$, a contradiction. If $\lim_{n\to\infty} m_{n+1}/m_n = 0$ or if this limit does not exist, the left side of (32) still has limit zero but the right side has no limit, another contradiction.

The treatment of the second class of polynomials obtained when $\lambda_2 = 2$ is also based upon the expressions for b_n and c_n in (28) and (29). It is clear from (28) that $b_n < 0$ for $n = 0,1,2,\ldots;$ and consequently from (11) it is only necessary to consider the case $b_1 < b_o$, for if $b_1 > b_o$, then $b_n > 0$ for n sufficiently large.

So suppose that $b_1 < b_o$. Since $c_n = k_n^2/m_n m_{n-1} > 0$, $n=1,2,3,\ldots$, the expression for b_n in (28) can be rewritten as

$$b_n = -\sqrt{c_n m_{n-1}/m_n} - \sqrt{c_{n+1}m_{n+1}/m_n} \ , \quad n = 0,1,2,\ldots,$$

and thus

$$b_n/n = -\left(\sqrt{c_n/n^2}\right)/\sqrt{m_n/m_{n-1}} - \left(\sqrt{c_{n+1}/(n+1)^2}\ \sqrt{m_{n+1}/m_n}\right)(n+1)/n\,, \tag{33}$$

$n = 1,2,3,\ldots$. Now from (11) $\lim_{n\to\infty} b_n/n = b_1 - b_o$ and $\lim_{n\to\infty} c_n/n^2 = (b_1 - b_o)^2/4$. With these observations it is possible to prove the following theorem.

<u>Theorem VI</u>. If m_o, m_1, k_o, k_1, k_2 are given and if $\lim_{n\to\infty} m_{n+1}/m_n \neq 1$, it is impossible, by adding springs and masses as shown in Figure 1, to construct a system of harmonic oscillators such that the resulting values for b_n and c_n in (28) and (29) satisfy (11) (with $b_1 < b_o$) for every positive integer n. Thus for a given set of values m_o, m_1, k_o, k_1, k_2, it is impossible to construct a system such that for every positive integer n the characteristic polynomial is -- apart from a linear change of variable -- a Laguerre polynomial in ω^2, <u>unless</u> $\lim_{n\to\infty} m_{n+1}/m_n = 1$.

<u>Proof</u>. Suppose otherwise. Then as shown above $\lim_{n\to\infty} c_n/n^2 = (b_1 - b_o)^2/4$, $\lim_{n\to\infty} b_n/n = b_1 - b_o$, and (33) must hold for every positive integer n. Now suppose that $\lim_{n\to\infty} m_{n+1}/m_n = L^2$, $L > 0$, $L \neq 1$. Then as $n \to \infty$ (33) yields

$$b_1 - b_o = -|b_1 - b_o|/2L - |b_1 - b_o|L/2\,.$$

Since $b_1 < b_o$ (the case $b_1 > b_o$ has already been eliminated), this identity implies that $L = 1$, a contradiction. If $L = 0$ or $L = +\infty$, the right side of (33) has no limit, another contradiction. If the sequence $\{m_{n+1}/m_n\}$ has no limit but does have a finite limit inferior, the arguments above do not apply; the one which follows was suggested by Dr. W. R. Smythe of the Georgia Institute of Technology. Let $A = \liminf \sqrt{m_n/m_{n-1}}$, and note that A must be non-negative since $m_n > 0$ for every positive integer n. Then there exists a subsequence $\left\{\sqrt{m_{n_k}/m_{n_k-1}}\right\}$ with limit A, and (33) must hold for the terms of this subsequence:

$$b_{n_k}/n_k = -(c_{n_k}/n_k^2)^{1/2}(m_{n_k}/m_{n_k-1})^{-1/2} \tag{34}$$
$$- [c_{n_k+1}/(n_k+1)^2]^{1/2}(m_{n_k+1}/m_{n_k})^{1/2}(n_k+1)/n_k .$$

As $n_k \to \infty$ (34) yields

$$2 = 1/A + \lim_{n_k\to\infty}\sqrt{m_{n_k+1}/m_{n_k}} \geq 1/A + A , \tag{35}$$

provided $A \neq 0$. If $A = 0$ the same contradiction arises as was noted above under the assumption that $L = 0$. From (35), with $A \neq 0$, it follows that $(A-1)^2 \leq 0$ and so $A = 1$. Now let $B = \limsup \sqrt{m_n/m_{n-1}}$, and note that $B \geq 1$ since $A = 1$. Denote a subsequence which converges to B by $\left\{\sqrt{m_{n_j+1}/m_{n_j}}\right\}$. Equation (33) must hold for the terms of this subsequence also, yielding

$$2 = 1/\lim_{n_j\to\infty}\sqrt{m_{n_j}/m_{n_j-1}} + B \geq 1/B + B \tag{36}$$

provided B is finite. In this case $B = 1$ from (36) and so $A = B$, which contradicts the assumption that the sequence $\{m_{n+1}/m_n\}$ has no limit. If B is not finite, the same contradiction will arise from (33) (with the terms of the subsequence which converges to B inserted) as was noted under the assumption above that $L = \infty$.

The following interesting example shows that the conclusion of Theorem VI is not valid if the hypothesis $\lim_{n \to \infty} m_{n+1}/m_n \neq 1$ is dropped. Construct a system as shown in Figure 1 with $m_n = 1$, $k_n = n$, $n = 0,1,2,\ldots$ ($k_o = 0$ merely indicates that the leftmost mass is not coupled to the left side of the rigid support). Then $b_n = -(2n + 1)$ $(n = 0,1,2,\ldots)$, $c_n = n^2$ $(n = 1,2,3,\ldots)$, and it is readily verified that the resulting characteristic polynomials are the Laguerre polynomials in ω^2.

The remaining cases to be investigated are those for which $\lambda_2 \neq 2$ -- i.e., the generalized Bessel and Jacobi polynomials. Consider the Bessel polynomials first. As was previously noted these are obtained provided the hypotheses of Theorem II are satisfied, $\lambda_2 \neq 2$, and

$$(\lambda_2 - 1)^2(b_o - b_1)^2 + 4\lambda_2(\lambda_2 - 2)c_1 = 0 \,. \tag{37}$$

Since $c_1 = k_1^2/m_1m_o > 0$, (37) implies that $0 < \lambda_2 < 2$ (λ_2 cannot be zero because λ_2 and λ_0 would then be equal).

Theorem VII. For a given set of initial values m_o, m_1, k_o, k_1, k_2, it is impossible, by adding springs and masses as shown in Figure 1, to construct a system of harmonic oscillators such that the resulting values of b_n and c_n in (28) and (29) satisfy the requirements of Theorem II

<u>and</u> (37). Thus for a given set of values m_o, m_1, k_o, k_1, k_2, it is impossible to construct a system such that for every positive integer n the characteristic polynomial is -- apart from a linear change of variable -- a Bessel polynomial in ω^2.

<u>Proof</u>: A physically realizable system must be such that b_n and c_n satisfy equations (9) with each $b_n < 0$ and each $c_n > 0$. The proof consists of showing that the condition $c_n > 0$ cannot be satisfied for every positive integer n. If the first of equations (9) is multiplied by b_n, an expression is obtained for $b_n b_{n+1}$ which when inserted into the second of equations (9) reduces it to

$$[(2n + 1)\lambda_2 - 4n]c_{n+1} - [(2n - 3)\lambda_2 - (4n - 8)]c_n \tag{38}$$

$$= (2 - \lambda_2)b_n^2 + [(b_o + b_1)\lambda_2 - (b_1 + 3b_o)]b_n$$

$$+ c_1\lambda_2 + b_o^2 - (\lambda_2 - 1)b_o b_1 .$$

The right side of (38) is a quadratic in b_n and in fact under the present conditions it is a perfect square. To see this it is only necessary to note that $(2 - \lambda_2) > 0$ and to compute the discriminant, which reduces somewhat surprisingly to the left side of (37). Since (37) holds by hypothesis, the assertion follows. Thus the right side of (38) is never negative; if its non-negative value is denoted by p_n, (38) may be rewritten

$$[(2n + 1)\lambda_2 - 4n]c_{n+1} = [(2n - 3)\lambda_2 - (4n - 8)]c_n + p_n . \tag{39}$$

For $n = 1$ (39) is

$$(3\lambda_2 - 4)c_2 = (4 - \lambda_2)c_1 + p_1 ,$$

and since c_1 and c_2 must be positive, it follows that $4/3 < \lambda_2$. Thus $4/3 < \lambda_2 < 2$. Now, inspired by the form of the coefficients in (39), construct the sequence $\{a_n\}$ where $a_n = 4n/(2n + 1)$, $n \geq 1$. Note that $a_1 = 4/3$ and $\lim_{n\to\infty} a_n = 2$. It is then evident that there is a positive integer k such that

$$(4k - 8)/(2k - 3) \leq \lambda_2 \leq 4k/(2k + 1) .$$

In fact, strict inequality holds, since the endpoints are special cases of the form which λ_2 cannot assume, as shown in Chapter III. Suppose that $c_k > 0$. Then the right side of (39) is positive for $n = k$, since $\lambda_2 > (4k - 8)/(2k - 3)$ and $p_k \geq 0$. But since $\lambda_2 < 4k/(2k + 1)$ the left side of (39) for $n = k$ cannot be positive unless $c_{k+1} < 0$. Thus the equations (9) cannot be satisfied since each c_n must be positive.

For the Jacobi polynomials no such theorem is possible. Not only does the simple case in which all the spring constants are equal and all the masses are equal yield Jacobi polynomials; there are other more interesting cases as well. In particular consider the system in Figure 1 again, with the masses and spring constants determined by $k_n = k_o r^n$, $m_n = m_o r^n$, where r is an arbitrary positive constant (yielding progressively stronger or weaker springs according as $r > 1$ or $r < 1$). Here $b_n = -(1 + r)k_o/m_o$ and $c_n = rk_o^2/m_o^2$. The hypotheses of Theorem II are satisfied in this case, and by means of the classification in Chapter III the characteristic polynomials are identified as Jacobi polynomials in ω^2.

Other physical problems give rise in a similar way to Jacobi polynomials. For example, the following sequence occurs in control system theory:*

$$\varphi_0 = 1$$

$$\varphi_1 = x$$

$$\varphi_{n+1} = x\varphi_n - \varphi_{n-1} .$$

Here each b_n equals zero; each c_n equals one; and it is easy to check that the hypotheses of Theorem II are again satisfied. The polynomials are solutions of

$$(x^2 - 4)y'' + 3xy' - n(n + 2)y = 0, \quad ' \sim d/dx ,$$

which becomes, after the change of variable $x = 2t$, the equation

$$(t^2 - 1)y'' + 3ty' - n(n + 2)y = 0, \quad ' \sim d/dt .$$

The only polynomial solution of this equation, for fixed n, is the Jacobi polynomial $P_n^{(\frac{1}{2}, \frac{1}{2})}(t)$.

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*See Problem 64-16, posed by N. Mullineux, in the section on Problems and Solutions, SIAM Review, Vol. 6, No. 4 (October, 1964), p. 455.

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VITA

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